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LETTER TO THE EDITOR

Critical point analysis of various fermionic field theories in the large N expansion

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Abstract. We compute the critical exponents corresponding to the anomalous dimensions of the fields and vertices in a model involving fermions with a four point interaction, coupled to a U(1) gauge field, at $O(1/N)$ in a large N expansion in arbitrary dimensions. We also discuss the QED Ward identity within this formalism.

The large N expansion has been widely used to examine the quantum properties of field theories with an internal symmetry. Instead of analysing the models in the low coupling limit (perturbation theory) one treats the parameter $1/N$ as a coupling constant with N large. Consequently, one can carry out a leading order analysis by renormalizing Green functions, using, say, dimensional regularization, and subsequently deducing the appropriate functions occurring in the renormalization group equation. However, in the conventional approach, it is not possible to go beyond the leading order due to the occurrence of intractable integrals. Instead one uses a different method, developed in [1, 2] for the bosonic $O(N)$ σ model and subsequently applied to other models in [3, 4]. Here, one analyses the models at the d-dimensional critical point of the theory, where the mass vanishes, which essentially allows one to compute the integrals which arise at the next order. One consequence is that information on the coefficients which appear in the renormalization group functions of perturbation theory can be determined to all orders, at the particular large N approximation.

Recently, the techniques of [1] have been applied to quantum electrodynamics (QED) in [4], where the electron anomalous dimension in the Landau gauge was deduced from a consistency equation at leading order, which is related to a critical point analysis of the Dyson equations. To go beyond this order in an analogous fashion to [2], one encounters a much more intricate and complicated analysis. As a first step in attempting such a calculation, we require an independent determination of the vertex anomalous dimension which occurs. Usually it is deduced at $O(1/N)$ by considering the bare consistency equation at $O(1/N^2)$ [1]. After a suitable regularization has been introduced, this quantity is defined in such a way that the finite consistency equation is independent of position or momentum variables. Further, rather than just consider QED, we will analyse a more general model, which, in addition to a U(1) gauge field coupled to fermions, includes a four fermi interaction. Such a model has recently been analysed in strictly three dimensions in [5], and so it is of interest to provide further results both beyond the leading order in large N and also to determine the critical exponents in *arbitrary* dimensions. Further, there has been recent interest in examining

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four fermi interactions near four dimensions within the large N expansion, both in models with this type of interaction and also in models where the fermions have additional types of coupling [6].

We begin by first deriving the consistency equation for η, the anomalous dimension of the fermion ψ^i, $1 \leq i \leq N$, where we use the (massless) Lagrangian

$$L = \mathrm{i}\bar{\psi}^i \not{\partial} \psi^i - \frac{(F_{\mu\nu})^2}{4e^2} + A_\mu \bar{\psi}^i \gamma^\mu \psi^i + \rho \bar{\psi}^i \psi^i - \frac{\rho^2}{2g^2}. \tag{1}$$

The field A_μ corresponds to a U(1) gauge field and ρ is an auxiliary scalar field which, if eliminated by its equation of motion, yields the four fermi interaction. The coupling constants, e and g, have been rescaled into the relevant kinetic terms as a first stage in the application of the methods of [1], and we have set $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. To fix notation, we introduce the asymptotic scaling functions for the propagators of each field, in momentum space, using the name of the field to denote the function

$$\psi(k) \sim \frac{A}{(k^2)^{\mu-\alpha}} \qquad \rho(k) \sim \frac{B}{(k^2)^{\mu-\beta}} \qquad A_{\mu\nu}(k) \sim \frac{C}{(k^2)^{\mu-\gamma}} \left[\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] \tag{2}$$

where A, B and C are the amplitudes of the functions and α, β and γ are the exponents of the respective fields. From a dimensional analysis of (1), [1, 4], they are defined by

$$\alpha = \mu + \tfrac{1}{2}\eta \qquad \beta = 1 - \eta - \chi_\rho \qquad \gamma = 1 - \eta - \chi_A \tag{3}$$

where $d = 2\mu$ is the dimension of spacetime, η is the (gauge-dependent) anomalous dimension of the fermion and χ_ρ and χ_A are the anomalous dimensions of the respective vertices of (1). As in [4, 7], we choose to calculate in the Landau gauge and have taken $A_{\mu\nu}(k)$ accordingly.

As a preliminary to computing χ_ρ and χ_A, we require not only an expression for η, but also equations for the amplitudes A, B and C. They are deduced by solving the skeleton Dyson equations with dressed propagators, truncated to the appropriate order in large N, at the non-trivial d-dimensional critical point of the theory [1]. The asymptotic scaling functions of the two point functions, which appear in the Dyson equations, are given by the inverses of (2) in momentum space, where $A_{\mu\nu}$ is inverted on the transverse subspace [7, 8]. Thus from the Dyson equations of figure 1, where we substitute (2) for the lines comprising the one loop graphs and carry out the simple integrations, we have respectively,

$$0 = 1 + \frac{z\nu(\mu-\alpha-1, \mu-\beta, \alpha+\beta)}{(\mu-\alpha-1)(\alpha+\beta)} + \frac{y\alpha(2\mu-1)(\gamma+1-\mu)}{(\mu-\gamma)(\alpha+\gamma)} \nu(\mu-\gamma, \mu-\alpha, \alpha+\gamma) \tag{4}$$

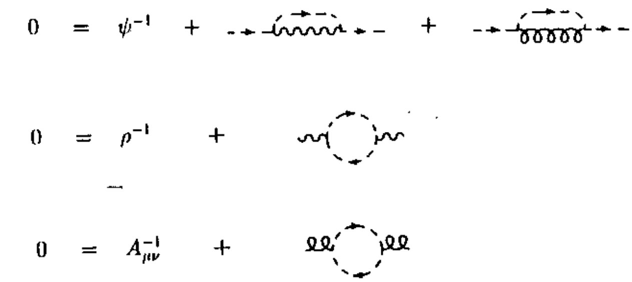

Figure 1. Skeleton Dyson equations with dressed propagators.

$$0 = 1 - \frac{zNt\alpha}{(\mu-\alpha-1)}\,\nu(\mu-\alpha-1, \mu-\alpha, 2\alpha+1) \tag{5}$$

$$0 = 1 + \frac{2yNt\alpha^2}{(\mu-\alpha-1)(2\alpha+1)}\,\nu(\mu-\alpha-1, \mu-\alpha, 2\alpha+1) \tag{6}$$

where $\nu(\alpha_1, \alpha_2, \alpha_3) = \pi^\mu \prod_{i=1}^3 a(\alpha_i)$, with $a(\alpha) = \Gamma(\mu-\alpha)/\Gamma(\alpha)$. Also we have set $z = A^2B$ and $y = A^2C$. The manipulation of the γ-matrices is performed purely with $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, which is valid in arbitrary dimensions and we use the convention $\mathrm{tr}\, 1 = t$, to allow for $t = 2$ or 4 depending on the dimension of the representation of the γ-matrices one chooses. First, eliminating y and z from (4)-(6) and using (3), we find

$$\eta = \frac{[4\mu^2 - 10\mu + 5]\Gamma(2\mu-1)}{\Gamma^3(\mu)\Gamma(2-\mu)Nt}. \tag{7}$$

In three dimensions, $\eta = -8/(Nt\pi^2)$, which agrees with the explicit result of [5], which is a check on our calculation. With (7), we obtain the following expressions for y and z,

$$\begin{aligned} y &= \frac{(2\mu-1)\Gamma(\mu)\eta}{2\pi^\mu[4\mu^2-10\mu+5]} \\ z &= -\frac{(\mu-1)\Gamma(\mu)\eta}{\pi^\mu[4\mu^2-10\mu+5]}. \end{aligned} \tag{8}$$

To deduce χ_ρ and χ_A, which we will require for checks on an $O(1/N^2)$ computation of η in both QED and (1), we adapt the approach of [9], which was applied to σ models on Grassmannian spaces in [10, 11]. In [9], the vertex anomalous dimensions are deduced directly from the 3-point functions rather than an $O(1/N^2)$ renormalization of the corrections to the η self-consistency equation. First, to regulate the divergences which will arise in computing the one loop graphs, we shift the exponents of both the ρ and A_μ fields by an amount Δ, i.e. $\beta \to \beta - \Delta$, $\gamma \to \gamma - \Delta$. Then from general arguments given in [9, 10], the full $\rho\bar{\psi}\psi$ vertex, for instance, at criticality is equivalent to $-\chi_\rho/(2\Delta)$. Thus, an expression for χ_ρ, to leading order, can be obtained by computing the leading order graphs *explicitly*, using (2), with the regularized exponents. The relevant graphs for χ_ρ and χ_A are given in figures 2 and 3, where to this order in $1/N$ there are no

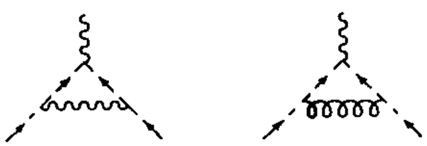

Figure 2. Corrections to $\rho\bar{\psi}\psi$ vertex.

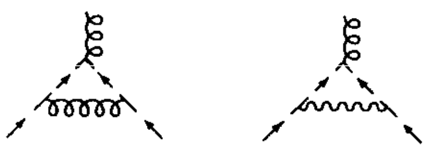

Figure 3. Corrections to $A_\mu\bar{\psi}\gamma^\mu\psi$ vertex.

three loop graphs due to Furry's theorem for gauge theories and its modification for the model (1). The graphs are analysed with (2), where a non-zero momentum, p, flows through the fermion legs only. Using, for instance, the chain rule of [2], and with $\alpha=\mu$, $\beta=\gamma=1$, figure 2 is equivalent to

$$[z+(2\mu-1)y]\nu(\mu-1,1,\mu-\Delta)]\left(\frac{m^2}{p^2}\right)^{\Delta}. \tag{9}$$

An arbitrary mass scale m enters at each vertex of (1) to take account of the change in dimension due to the introduction of the regulator [9]. Only the residues of the poles with respect to Δ in (9) contribute to χ_ρ. Thus,

$$\chi_\rho=-\frac{(4\mu^2-6\mu+3)}{(4\mu^2-10\mu+5)}\eta \tag{10}$$

and when $d=3$, $\chi_\rho=-24/(Nt\pi^2)$, which agrees with [5].

Similarly, χ_A is deduced by considering the graphs of figure 3, where the pole part has to be equivalent to $-\gamma^\sigma\chi_A/(2\Delta)$. The γ-matrix arises, of course, due to Lorentz symmetry. The second graph of figure 3 is easy to compute, whilst the first can be simplified by using

$$\int_k \frac{k_\mu k_\nu}{(k^2)^\alpha((k-p)^2)^\beta} =\frac{(\alpha+\beta-\mu-2)\nu(\alpha-1,\beta-1,2\mu-\alpha-\beta+2)}{2(\alpha-1)(\beta-1)(p^2)^{\alpha+\beta-\mu-1}} \times\left[\eta_{\mu\nu}+\frac{2(\mu-\alpha+1)(\alpha+\beta-\mu-1)}{(\mu-\beta)}\frac{p_\mu p_\nu}{p^2}\right] \tag{11}$$

for arbitrary values of α and β. Of course, in calculating the first graph of figure 3, it will have the structure $P\gamma^\sigma+Q\not{p}\gamma^\sigma\not{p}/p^2$ from general considerations, where P and Q are dimensionless quantities depending only on μ, Δ and the exponents. Therefore, we must ensure that there are no pole pieces arising in the second term, Q, which would otherwise spoil the renormalizability of the theory [10]. The explicit calculation, with $\alpha=\mu$, $\beta=\gamma=1$, gives

$$\frac{z(1-\Delta)}{(\mu-1+\Delta)}\nu(\mu-1+\Delta,1,\mu+1-\Delta)\left(\frac{m^2}{p^2}\right)^{\Delta} \times\bigg[(\Delta-1)\gamma^\sigma[(\mu-1+\Delta)(2(\mu-1)^2-(\mu-\Delta))+\Delta(\mu-\Delta)] -\frac{\gamma^\mu\not{p}\gamma^\sigma\not{p}\gamma_\mu\Delta}{2p^2}[2(\mu-1+\Delta)(\mu-1)-(\mu-\Delta)]\bigg]. \tag{12}$$

Thus it is easy to see there is no divergent contribution to Q. Adding the analogous expression from the second graph, the pole pieces with respect to Δ rearrange to give $\gamma^\sigma\eta_1/(2\Delta N)$, where $\eta=\Sigma_{i=1}^\infty\eta_i/N^i$. Thus,

$$\chi_A=-\eta \tag{13}$$

at this order in $1/N$ for (1).

If one considered QED only and ignored the second interaction of (1), then we would have

$$\chi_A^{\mathrm{QED}}=-\eta^{\mathrm{QED}} \tag{14}$$

where [4]

$$\eta_1^{\mathrm{QED}} = -\frac{(2\mu-1)(2-\mu)\Gamma(2\mu)}{4\Gamma^2(\mu)\Gamma(\mu+1)\Gamma(2-\mu)}. \tag{15}$$

The result (14) is consistent with the Ward identity of QED. More precisely, in (1) we have absorbed the coupling constant, e, into the definition of the gauge field to ensure the vertices have unit coupling, required by the method [1]. If one were to carry out an explicit perturbative renormalization of QED, then denoting bare quantities with a subscript 0, A_0^μ is related to the renormalized field A^μ by

$$A_0^\mu = \frac{Z_1}{Z_2} A^\mu \tag{16}$$

where we use the conventional notation for the renormalization constants Z_i. (See, for example, [12].) However, the Ward identity implies $Z_1 = Z_2$ to all orders in perturbation theory and therefore there is no wavefunction renormalization of the field A_μ of QED, in our definition of the gauge field. In other words, A_μ has zero anomalous dimension, which, from (3), implies $\chi_A^{\mathrm{QED}} = -\eta^{\mathrm{QED}}$. In determining this explicitly at leading order, (14), which provides us with a check on the computation, we have verified the Ward identity holds to all orders within this large N approximation. Including the four fermi interaction, the model (1) has a similar property.

We conclude by noting that we have solved (1) at leading order in large N by giving analytic expressions for the critical exponents of the fields of the model (1), in arbitrary dimensions. Further, the formalism which has been introduced will serve as a basis for going beyond this leading order to calculate the $O(1/N^2)$ corrections to various exponents both for QED and the model (1), which we hope to return to later.

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